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A Class of Approximate Solutions to Linear Operator Equations*

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A certain class of approximate solutions to linear operator equations is studied, in which the domain and range of the operator are both Hilbert spaces possessing continuous reproducing kernels. The broad class of operators considered here includes integral, differential, and integrodifferential operators. The results are applied to obtain approximate solutions and related (favorable) convergence rates for two-point boundary-value problems and associated integrodifferential equations.

1. SUMMARY

We consider a class of approximate solutions to linear operator equations where the domain and range of the operator are both Hilbert spaces possessing continuous reproducing kernels. The (broad) class of operators considered here includes integral, differential, and integrodifferential operators. The specialization to Fredholm integral equations of the first kind has been considered in detail in [5]. The main convergence theorem has been proved there.

The purpose of this article is to reformulate the approximate solutions and convergence results of [5] in a more general framework. Then these results are applied to obtain approximate solutions and related convergence rates for two-point boundary value problems and associated integrodifferential equations.

We note that there is an interesting history of the use of reproducing kernel Hilbert spaces to solve problems in approximation theory. See, for example Golomb and Weinberger [2], and, especially, Ciarlet and Varga [1] who consider approximate solutions to differential equations. However, it is

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believed that the approximate solutions described here for boundary value problems are new, in the generality discussed here. The approximate solutions we study are exact on a certain n-dimensional subspace which may be identified.

In Section 2 we give the approximate solutions and convergence results, restated from [5] in the context of general linear operator equations. The properties of reproducing kernel spaces that we use here are stated briefly in Section 2. For more details the reader may see [5] and references there. In Section 3 the results of Section 2 are applied to the approximate solution of 2-point boundary value problems. Section 4 gives an example to show what the method is doing and to indicate that the convergence rates for the method given here cannot be improved upon. The method, applied to $L_m f = g, f \in \mathscr{B}$, where L_m is an m-th order linear differential operator, and $\mathscr{B}$ is an appropriate set of boundary conditions, is equivalent to the following: g is interpolated at n values of the ordinate by a linear combination of suitably chosen functions, to obtain an approximation $\hat{g}$. The approximate solution $\hat{f}$, then, satisfies exactly $L_m \hat{f} = \hat{g}, \hat{f} \in \mathscr{B}$. Section 5 gives the application to linear integrodifferential boundary value problems.

2. Properties of Reproducing Kernel Spaces. The Approximate Solutions and Their Convergence Rates

Let $\mathscr{H}_R$ be a Hilbert space possessing a (real) reproducing kernel $R(s, s')$, $s, s' \in S$, where S is a closed bounded interval of the real line. By the properties of reproducing kernels, the function R_s defined by

$$R_s(\cdot) = R(s, \cdot) \tag{2.1}$$

is in $\mathscr{H}_R$ and

$$\langle f, R_s \rangle_R = f(s), \; s \in S, f \in \mathscr{H}_R\,, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle_R$ is the inner product in $\mathscr{H}_R$. Let N be any continuous linear functional on $\mathscr{H}_R$. Then its representer $\eta(\cdot)$, is given by the following formula:

$$Nf = \langle \eta, f \rangle_R\,; \quad \eta(s) = \langle \eta, R_s \rangle_R = NR_s\,. \tag{2.3}$$

Let T be a closed, bounded interval of the real line. We consider operators K defined from $\mathscr{H}_R$ into the real-valued functions on T of the form

$$Kf = g \tag{2.4}$$

$$(Kf)(t) = g(t) = \langle \eta_t\,, f \rangle_R\,, \; t \in T$$

where $\eta_t \in \mathscr{H}_R$, $t \in T$. That is, K is required only to have the property that the linear functionals $\{N_t, t \in T\}$ defined by

$$N_t f = (Kf)(t), \qquad t \in T \tag{2.5}$$

are all continuous in $\mathscr{H}_R$. Given K with this property, η_t is found by

$$\eta_t(s) = \langle \eta_t, R_s \rangle_R = (KR_s)(t). \tag{2.6}$$

Let V be the closure of the span of $\{\eta_t, t \in T\}$, in $\mathscr{H}_R$. Then the null space of K in $\mathscr{H}_R$ is $V^\perp$, that is,

$$\langle \eta_t, f \rangle_R = 0,\ t \in T, f \in \mathscr{H}_R \Rightarrow f \in V^\perp. \tag{2.7}$$

Let $\Delta = \{t_1, t_2, ..., t_n\}$, where $t_1 < t_2 < \cdots < t_n$, $[t_1, t_n] = T$. We let the (n-th) approximate solution $\hat{f} \in \mathscr{H}_R$ to the equation

$$Kf = g$$

be that element of minimum $\mathscr{H}_R$-norm which satisfies

$$(Kf)(t) = \langle \eta_t, f \rangle_R = g(t),\ t \in \Delta. \tag{2.8}$$

If f is any element in $\mathscr{H}_R$ satisfying (2.8), then $\hat{f}$ is the projection, $P_{V_n} f$, of f onto the subspace V_n of V spanned by $\{\eta_t, t \in \Delta\}$. Let $Q(t, t')$ be the non-negative definite kernel on $T \times T$ given by

$$Q(t, t') = \langle \eta_t, \eta_{t'} \rangle_R . \tag{2.9}$$

If $\{\eta_t, t \in T\}$ are linearly independent, then the $n \times n$ matrix Q_n with i, j-th entry $Q(t_i, t_j)$, $t_i, t_j \in \Delta$ is strictly positive definite, and we may write $\hat{f}(s)$ explicitly as

$$\hat{f}(s) = (P_{V_n} f)(s) = (\eta_{t_1}(s), \eta_{t_2}(s), ..., \eta_{t_n}(s))\, Q_n^{-1}(g_1, g_2, ..., g_n)' \tag{2.10}$$

where $g_i = g(t_i)$, $t_i \in \Delta$. In the remainder of this paper it will be assumed that $\{\eta_t, t \in T\}$ are linearly independent. It may be shown that

$$Q(t, t') = \langle \eta_t, \eta_{t'} \rangle_R = N_t N_{t'} R(\cdot, \cdot) \tag{2.11}$$

where N_t is defined by (2.5) and is applied to R considered as a function of the first argument, and $N_{t'}$ is applied to R as a function of the second argument. To see this, note that, for any reproducing kernel Hilbert space, the family $\{R_s, s \in S\}$ span $\mathscr{H}_R$. Then let $\eta_t^{(l)}, \eta_{t'}^{(l)}$ be the l-th members in two Cauchy sequences tending to η_t and $\eta_{t'}$ respectively,

$$\begin{aligned} \eta_t^{(l)} &= \sum_{i=1}^{l} c_{ilt} R_{s_{il}}, \\ \eta_{t'}^{(l)} &= \sum_{i=1}^{l} c_{ilt'} R_{s_{il}}, \end{aligned} \tag{2.12}$$

and use the fact that $\langle R_s, R_{s'}\rangle_R = R(s, s')$ and hence

$$\langle \eta_t^{(l)}, \eta_{t'}^{(l)}\rangle_R = \sum_{i=1}^{l}\sum_{j=1}^{l} c_{ilt}c_{jlt'}R(s_{il}, s_{jl}). \tag{2.13}$$

Suppose that $Q(t, t')$ is continuous for $(t, t') \in T \times T$, then $\{\eta_t, t \in T, t \text{ rational}\}$ is dense in the set $\{\eta_t, t \in T\}$. Let P_V be the projection operator in $\mathscr{H}_R$ onto V and let

$$\| \Delta \| = \max_i (t_{i+1} - t_i). \tag{2.14}$$

Then it follows that

$$\lim_{\|\Delta\|\to 0} \| P_V f - P_{V_n} f\|_R = 0 \tag{2.15}$$

for any fixed $f \in \mathscr{H}_R$. Obviously we have no information from g concerning $f - P_V f \in V^\perp$. To study $| P_V f(s) - P_{V_n} f(s)|$ we use the inequalities

$$\begin{aligned} | P_V f(s) - P_{V_n} f(s)| &= |\langle (P_V - P_{V_n})f, R_s\rangle_R |, \\ &= |\langle (P_V - P_{V_n})f, (P_V - P_{V_n})R_s\rangle_R |, \\ &\leqslant \| P_V f - P_{V_n} f\|_R \| P_V R_s - P_{V_n} R_s \|_R . \end{aligned} \tag{2.16}$$

Let $\mathscr{H}_Q$ be the reproducing kernel Hilbert space with reproducing kernel $Q(t, t')$ given by (2.11). ($\mathscr{H}_Q$ always exists uniquely for positive definite Q). Let Q_t be the element of $\mathscr{H}_Q$ defined by

$$Q_t(\cdot) = Q(t, \cdot). \tag{2.17}$$

Let $\langle \cdot, \cdot \rangle_Q$ be the inner product in $\mathscr{H}_Q$. Since $\{Q_t, t \in T\}$ span $\mathscr{H}_Q$, and $\{\eta_t, t \in T\}$ span V, and

$$\langle \eta_t, \eta_{t'}\rangle_R = Q(t, t') = \langle Q_t, Q_{t'}\rangle_Q \tag{2.18}$$

there is an isometric isomorphism between V and $\mathscr{H}_Q$ generated by the correspondence

$$\eta_t \in V \sim Q_t \in \mathscr{H}_Q . \tag{2.19}$$

Then $f \in V \sim g \in \mathscr{H}_Q$ if and only if

$$\langle \eta_t, f\rangle_R = g(t) = \langle Q_t, g\rangle_Q, \quad t \in T. \tag{2.20}$$

In other words, $f \in V \sim g \in \mathscr{H}_Q$ if and only if

$$g(t) = (Kf)(t), \quad t \in T. \tag{2.21}$$

Thus the range $K(\mathscr{H}_R)$ of K is $\mathscr{H}_Q$, and K restricted to V is a 1 : 1 invertible operator from V to $\mathscr{H}_Q$.

To discuss rates of convergence of the right-hand side of (2.16) it is convenient to perform the calculations in $\mathscr{H}_Q$ and make use of the isometric isomorphism generated by (2.19). To this end we list the following table of corresponding elements and sets, where the entries on the left are in $\mathscr{H}_R$.

$$\begin{array}{rll} V \sim \mathscr{H}_Q & & \\ f \sim g & g(t) = \langle \eta_t, f\rangle_R, \quad t \in T & \\ \eta_t \sim Q_t & & \\ V_n \sim T_n & T_n = \operatorname{span}\{Q_t, t \in \Delta\} & \\ P_V R_s \sim \gamma_s & \gamma_s(t) = \langle \eta_t, P_V R_s\rangle_R = \langle \eta_t, R_s\rangle_R = \eta_t(s). & \end{array} \tag{2.22}$$

If the linear functional D_s^ν defined, for fixed s, by

$$D_s^\nu f = f^{(\nu)}(s) \tag{2.23}$$

is continuous in $\mathscr{H}_R$, then it has the representer R_s^ν defined by

$$\langle R_s^\nu, f\rangle_R = f^{(\nu)}(s), f \in \mathscr{H}_R,$$

where, by (2.3),

$$R_s^\nu(s') = D_s^\nu R_{s'} = (\partial^\nu/\partial s^\nu)\, R(s', s). \tag{2.24}$$

D_s^ν is continuous if $R_s^\nu \in \mathscr{H}_R$. If $R_s^\nu \in \mathscr{H}_R$, then

$$R_s^\nu \sim \gamma_s^\nu, \tag{2.25}$$

where

$$\gamma_s^\nu(t) = (\partial^\nu/\partial s^\nu)\, \gamma_s(t). \tag{2.26}$$

A proof of (2.26), for $\nu = 1$, proceeds by noting that

$$P_V R_s^1 = \lim_{\epsilon\to 0} (1/\epsilon)(P_V R_{s+\epsilon} - P_V R_s) \sim \lim_{\epsilon\to 0} (1/\epsilon)(\gamma_{s+\epsilon} - \gamma_s) = \gamma_s^1 \tag{2.27}$$

where the limits are taken in the strong topology of V and $\mathscr{H}_Q$. Let P_{T_n} be the projection operator in $\mathscr{H}_Q$ onto the subspace T_n of (2.22). Thus, if $R_s^\nu \in \mathscr{H}_R$, by the isometric isomorphism of (2.19)

$$\begin{aligned} |(d^\nu/ds^\nu)(P_V f)(s) - (d^\nu/ds^\nu)(P_{V_n} f)(s)| &= |\langle P_V f - P_{V_n} f, R_s^\nu\rangle_R|, \\ &= |\langle g - P_{T_n} g, \gamma_s^\nu\rangle_Q|, \\ &= |\langle g - P_{T_n} g, \gamma_s^\nu - P_{T_n}\gamma_s^\nu\rangle_Q|, \\ &\leqslant \| g - P_{T_n} g\|_Q \|\gamma_s^\nu - P_{T_n}\gamma_s^\nu\|_Q, \end{aligned} \tag{2.28}$$

where $g = Kf$.

Some of the convergence properties of the approximate solution (2.10) to Eq. (2.4) may be obtained by the following theorem, proved in [5, Eq. (2.19)].

THEOREM 1. *Suppose that $Q(t, t')$ satisfies*:

(i) $(\partial^l/\partial t^l)\, Q(t, t')$ *exists and is continuous on* $T \times T$ (2.29)

for $t \neq t'$, $l = 0, 1, 2,..., 2q$, $(\partial^l/\partial t^l)\, Q(t, t')$ *exists and is continuous on* $T \times T$ *for* $l = 0, 1, 2,..., 2q - 2$;

(ii) $\lim_{t \uparrow t'} (\partial^{2q-1}/\partial t^{2q-1})\, Q(t, t')$ *and* $\lim_{t \downarrow t'} (\partial^{2q-1}/\partial t^{2q-1})\, Q(t, t')$ (2.30)

exist and are bounded for all $t' \in T$.

and suppose that h has a representation

(iii) $h(t) = \int_T Q(t, t')\, \rho(t')\, dt'$ (2.31)

for some $\rho \in \mathscr{L}_2[T]$.

Then $h \in \mathscr{H}_Q$ *and*

$$\| h - P_{T_n} h \|_Q = O(\| \Delta \|^q). \tag{2.32}$$

When studying the case K is a differential operator, it will be convenient to use the following theorem.

THEOREM 2. *Let Q satisfy the hypotheses* (i) *and* (ii) *of Theorem* 1. *Then, for each* $t \in T$,

$$\| Q_t - P_{T_n} Q_t \|_Q = O(\| \Delta \|^{q-1/2}). \tag{2.33}$$

Theorem 2 is implicit in the proof of Theorem 1 in [5] and is a direct consequence of Eq. (2.36) of [5].

3. APPLICATION TO THE APPROXIMATE SOLUTION OF 2-POINT BOUNDARY VALUE PROBLEMS

Consider the problem

$$L_m f = g, f \in \mathscr{B} \tag{3.1}$$

where

$$L_m f(t) = \sum_{j=0}^{m} a_{m-j}(t) f^{(j)}(t) \qquad t \in T = [0, 1] = S$$

and we assume that $f \in C^{r-1}$, $f^{(r)} \in \mathscr{L}_2[0, 1]$, $a_0(t) \geqslant \delta > 0$ and

$$a_j \in C^{\max(2m, 2(r-m))}, j = 0, 1, 2,..., m.$$

$$\mathscr{B} = \{f: U_\nu f = \omega_\nu\,, \nu = 1, 2,..., m\},$$

$$U_\nu f = \sum_{j=0}^{m-1} \theta_{\nu j} f^{(j)}(0) + \sum_{j=0}^{m-1} \xi_{\nu j} f^{(j)}(1), \tag{3.2}$$

where the $\{U_\nu\}_{\nu=1}^m$ are linearly independent. Without loss of generality, we will take $\omega_\nu = 0$, $\nu = 1, 2,..., m$ in (3.2). We have $g \in C^{q-1}$, $g^{(q)} \in \mathscr{L}_2[0, 1]$, $q = r - m$.

We seek an approximate solution $\hat{f}$ in a Hilbert space $\mathscr{H}_R$ of functions

$$\mathscr{H}_R = \{f : f \in C^{r-1}, f^{(r)} \in \mathscr{L}_2[0, 1], f \in \mathscr{B}\}. \tag{3.3a}$$

Suppose that

$$\mathscr{H}_{\mathbf{R}} = \{f : f \in C^{r-1}, f^{(r)} \in \mathscr{L}_2[0, 1]\}^1 \tag{3.3b}$$

with reproducing kernel $\mathbf{R}(s, s')$, $s, s' \in S$, and U_ν, $\nu = 1, 2,..., m$ are continuous linear functionals in $\mathscr{H}_{\mathbf{R}}$. Then $\mathscr{H}_R$ is the subspace of $\mathscr{H}_{\mathbf{R}}$ of codimension m whose elements all satisfy the boundary conditions. If the reproducing kernel $\mathbf{R}(s, s')$ for $\mathscr{H}_{\mathbf{R}}$ is given, then the reproducing kernel $R(s, s')$ for this subspace may be found as follows. Let

$$\phi_\nu(s) = U_\nu \mathbf{R}_s = \sum_{j=0}^{m-1} \theta_{\nu j} \frac{\partial^\nu}{\partial t^\nu} \mathbf{R}(s, t)\Big|_{t=0} + \sum_{j=0}^{m-1} \xi_{\nu j} \frac{\partial^\nu}{\partial t^\nu} \mathbf{R}(s, t)\Big|_{t=1} \tag{3.4}$$

where

$$\mathbf{R}_s(s') = \mathbf{R}(s, s').$$

Let $\langle \cdot , \cdot \rangle_{\mathbf{R}}$ be the inner product in $\mathscr{H}_{\mathbf{R}}$, and let A be the $m \times m$ (positive definite) matrix with μ, ν-th entry $a_{\mu\nu}$,

$$a_{\mu\nu} = \langle \phi_\mu , \phi_\nu \rangle_{\mathbf{R}} = U_{\mu(s)} U_{\nu(s')} \mathbf{R}(s, s') \tag{3.5}$$

where $U_{\mu(s)}$ means the linear functional applied to the function with argument s. Then

$$\begin{aligned} R(s, s') &= \mathbf{R}(s, s') - \sum_{\mu,\nu=1}^{m} \phi_\mu(s)\, a^{\mu\nu} \phi_\nu(s'), \\ A^{-1} &= \{a^{\mu\nu}\}. \end{aligned} \tag{3.6}$$

It may be verified that $R(\cdot, s)$ and $R(s, \cdot) \in \mathscr{B}$ for each fixed s. Equation (3.6) may be verified by letting P_ϕ be the projection operator in $\mathscr{H}_{\mathbf{R}}$ onto the subspace spanned by $\{\phi_\nu\}_{\nu=1}^m$. Then, we must have

$$R(s, s') = \langle R_s , R_{s'} \rangle_R = \langle R_s , R_{s'} \rangle_{\mathbf{R}} = \langle \mathbf{R}_s - P_\phi \mathbf{R}_s , \mathbf{R}_{s'} - P_\phi \mathbf{R}_{s'} \rangle_{\mathbf{R}} . \tag{3.7}$$

The approximate solution $\hat{f}(s)$ is then that element of minimum $\mathscr{H}_R$ norm satisfying

$$L_m \hat{f}(t) = g(t), \quad t \in \Delta \tag{3.8}$$

[1] Examples of $\mathscr{H}_{\mathbf{R}}$ and associated inner product may be found in [4] and [5]. A slightly specialized case will be found in Section 4.

where $\hat{f}(t)$ is given by (2.10) with

$$\begin{aligned}\eta_t(s) &= L_m R_s(t),\\ &= \sum_{j=0}^{m} a_{m-j}(t) \frac{\partial^j}{\partial t^j} R(s, t),\end{aligned} \tag{3.9}$$

and

$$Q(s, t) = \langle \eta_s, \eta_t \rangle_R = \sum_{j=0}^{m} \sum_{k=0}^{m} a_{m-j}(s)\, a_{m-k}(t) \frac{\partial^{j+k}}{\partial s^j\, \partial t^k} R(s, t). \tag{3.10}$$

For the examples of **R** given in [4], $\{\eta_t, t \in [0, 1]\}$ of (3.9) are always linearly independent in $\mathscr{H}_R$ and $Q(t, t')$ is strictly positive definite. See [3, Theor. 8.1, p. 547]. Here

$$\mathscr{H}_Q = \{g : g \in C^{r-m-1}, g^{(r-m)} \in \mathscr{L}_2[0, 1]\}.$$

We remark on some properties of the approximate solution (2.10) with $\eta_t(s)$ and $Q(t, t')$ given by (3.9) and (3.10).

Let $\hat{g} = L_m \hat{f}$. Then, since

$$(L_m \eta_t)(s) = Q_t(s) \tag{3.11a}$$

we have

$$\hat{g}(s) = (Q_{t_1}(s), Q_{t_2}(s), ..., Q_{t_n}(s))\, Q_n^{-1} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} \tag{3.11b}$$

where $g_i = g(t_i)$. Note that $\hat{g}$ is the solution to the problem: Find $\hat{g} \in \mathscr{H}_Q$ to minimize $\| \hat{g} \|_Q$ subject to $\hat{g}(t_i) = g_i$, $i = 1, 2, ..., n$, and, if g is any element in $\mathscr{H}_Q$ with $g(t_i) = g_i$, then $\hat{g} = P_{T_n} g$.

Thus, the approximate solution satisfies

$$(L_m \hat{f})(t_i) - g(t_i) = \langle Q_{t_i}, L_m \hat{f} - g \rangle_Q = 0, \qquad i = 1, 2, ..., n, \tag{3.12}$$

demonstrating features of both collocation and Galerkin methods.

For any $g \in \mathscr{H}_Q$, $\hat{g}$ is the orthogonal projection in $\mathscr{H}_Q$ of g onto the n-dimensional subspace T_n spanned by $\{Q_t, t \in \Delta\}$. Thus, the method is *exact* if $g \in T_n$, or equivalently, if $f \in V_n$.

We may now apply the results given in Section 2 to the approximate solution $\hat{f}(s)$ of Eq. (3.1), where $\hat{f}(s)$ is given by (2.10), $\eta_t(s)$ and $Q(t, t')$ are defined by (3.9) and (3.10), and $R(s, s')$ is given by (3.6) with $\mathbf{R}(s, s')$ chosen as in (3.3b). If $\mathscr{H}_R$ is as in (3.3a) then the assumptions on a_j guarantee that $Q(t, t')$ satisfy the hypotheses (i) and (ii) of Theorem 1, with $q = r - m$.

THEOREM 3. *Let* $f \in \mathscr{H}_R$, *or, equivalently,* $g \in \mathscr{H}_Q$, *where* $Q(t, t')$ *given by* (3.10) *satisfies the hypotheses* (i) *and* (ii) *of Theorem* 1. *Then*

$$|f^{(\nu)}(s) - \hat{f}^{(\nu)}(s)| = O(\| \Delta \|^q), \nu = 0, 1, 2, ..., m-1, \tag{3.13}$$

$$|f^{(m)}(s) - \hat{f}^{(m)}(s)| = O(\| \Delta \|^{q-1/2}). \tag{3.14}$$

If g has a representation

$$g(t) = \int_0^1 Q(t, t') \rho(t') \, dt' \tag{3.15}$$

for some $\rho \in \mathscr{L}_2[0, 1]$, then

$$|f^{(\nu)}(s) - \hat{f}^{(\nu)}(s)| = O(\| \Delta \|^{2q}), \qquad \nu = 0, 1, 2,..., m-1, \tag{3.16}$$

$$|f^{(m)}(s) - \hat{f}^{(m)}(s)| = O(\| \Delta \|^{2q-1/2}). \tag{3.17}$$

Remark. Condition (3.15) entails that $g \in C^{2q-1}$, $g^{(2q)} \in \mathscr{L}_2$.

Proof of Theorem 3. First, we note that $L_m f = 0, f \in \mathscr{H}_R \Rightarrow f = 0$, since $\mathscr{H}_R \subset \mathscr{B}$. Thus $V = \mathscr{H}_R$. By the assumptions on the differential operator, there exists a Green's function $G_m(t, u)$ such that

$$f(s) = \int_0^1 G_m(s, u) \, g(u) \Rightarrow L_m f = g, \qquad f \in \mathscr{B} \tag{3.18}$$

and such that $\rho_s^\nu(u)$ defined by

$$\rho_s^\nu(u) = (\partial^\nu/\partial s^\nu) \, G_m(s, u) \qquad \nu = 0, 1, 2,..., m-1, \tag{3.19}$$

is a piecewise continuous function of u for each fixed s. We wish to apply (2.32) to the right-hand side of (2.28), with γ_s^ν of (2.28) satisfying hypothesis (iii) of Theorem 1. γ_s^ν is the element in $\mathscr{H}_Q$ corresponding to R_s^ν under the isomorphism (2.19).

To obtain a formula for γ_s we note that, for $L_m f = g, f \in \mathscr{H}_R, f \sim g$ and

$$\langle \gamma_s, g \rangle_Q = \langle R_s, f \rangle_R = f(s) = \int_0^1 G_m(s, u) \, g(u) \, du. \tag{3.20}$$

Therefore,

$$\gamma_s(t) = \langle \gamma_s, Q_t \rangle_Q = \int_0^1 G_m(s, u) \, Q_t(u) \, du = \int_0^1 G_m(s, u) \, Q(t, u) \, du \tag{3.21}$$

and, by differentiating $1, 2,..., m-1$, times with respect to s,

$$\gamma_s^\nu(t) = \int_0^1 Q(t, u) \, \rho_s^\nu(u) \, du \qquad \nu = 0, 1, 2,..., m-1. \tag{3.22}$$

Thus, γ_s^ν has a representation of the form (2.31), and hence (2.32) holds, giving

$$\| \gamma_s^\nu - P_{T_n}\gamma_s^\nu \|_Q = O(\| \Delta \|^q), \qquad \nu = 0, 1, 2,..., m - 1. \tag{3.23}$$

To study γ_s^m, note that

$$(L_m f)(t) = \left\langle \sum_{\nu=0}^{m} a_{m-\nu}(t)\, R_t^\nu, f \right\rangle_R = \langle Q_t, g\rangle_Q = g(t), \qquad t \in T, \tag{3.24}$$

so that

$$\sum_{\nu=0}^{m} a_{m-\nu}(t)\, R_t^\nu \sim Q_t \tag{3.25}$$

under the isomorphism of (2.19). But $R_s^\nu \sim \gamma_s^\nu$ so we must have

$$\sum_{\nu=0}^{m} a_{m-\nu}(s)\, \gamma_s^\nu = Q_s \tag{3.26}$$

or,

$$\gamma_s^m = \frac{1}{a_0(s)} Q_s - \sum_{\nu=0}^{m-1} \frac{a_{m-\nu}(s)}{a_0(s)} \gamma_s^\nu.$$

Now

$$\begin{aligned} &\| \gamma_s^m - P_{T_n}\gamma_s^m \|_Q \\ &\quad = \left\| \frac{1}{a_0(s)} (Q_s - P_{T_n}Q_s) - \sum_{\nu=0}^{m-1} \frac{a_{m-\nu}(s)}{a_0(s)} (\gamma_s^\nu - P_{T_n}\gamma_s^\nu) \right\|_Q \\ &\quad \leqslant (m+1)^{1/2} \left\{ \frac{1}{a_0^2(s)} \| Q_s - P_{T_n}Q_s \|_Q^2 + \sum_{\nu=0}^{m-1} \frac{a_{m-\nu}^2(s)}{a_0^2(s)} \| \gamma_s^\nu - P_{T_n}\gamma_s^\nu \|_Q^2 \right\}^{1/2} \\ &\quad = O(\| \Delta \|^{q-1/2}), \end{aligned} \tag{3.27}$$

by (2.33) and the assumptions on the coefficients $a_\nu(s)$.

Applying (3.23) and (3.27) to (2.28) gives the result.

4. Example

In this section we give a simple example, in an attempt to give the reader a feel for what the method is doing. In this example, we show that the convergence rates of (3.13), (3.14), (3.16), and (3.17) cannot be improved. Let $m = 2$, $\mathscr{B} : \{f(0) = f(1) = 0\}$.

Let

$$R(u, v) = \int_0^1 G(u, x)\, G(v, x)\, dx + \phi(u)\, \phi(v) \tag{4.1}$$

where

$$\begin{aligned} G(u, x) &= (-(1-u)/2)(u^2 - x^2) - (u/2)(1-x)^2, \qquad & u > x, \\ &= \qquad\qquad\qquad\qquad\quad - (u/2)(1-u)^2, \qquad & u < x, \end{aligned}$$

and

$$\phi(u) = -u(1-u)/2. \tag{4.2}$$

$G(u, x)$ is the Green's function for the problem $D^3 f = g, f(0) = f(1) = f''(0) = 0$. $\mathscr{H}_R$ is the Hilbert space $\{f : f(0) = f(1) = 0,\ f''$ absolutely continuous, $f''' \in \mathscr{L}_2[0, 1]\}$ with inner product

$$\langle f_1, f_2 \rangle_R = \int_0^1 f_1'''(u) f_2'''(u)\, du + f_1''(0) f_2''(0). \tag{4.3}$$

ϕ is that function which satisfies $\phi(0) = \phi(1) = 0$, $\phi''(0) = 1$, $\phi'''(u) = 0$. The choice of the boundary condition $f''(0) = 0$ in the selection of the Green's function and the concomitant choice of ϕ satisfying $\phi''(0) = 1$ is arbitrary. Here $r = 3$ and $q = 1$. Let $L_m f = f''$. Then

$$Q(s, t) = (\partial^4/\partial s^2 \partial t^2)\, R(s, t) = \min(s, t) + 1 \tag{4.4}$$

and $\mathscr{H}_Q$ is the Hilbert space

$$\{g : g \text{ absolutely continuous, } g' \in \mathscr{L}_2[0, 1]\} \tag{4.5}$$

with inner product

$$\langle g_1, g_2 \rangle_Q = \int_0^1 g_1'(s)\, g_2'(s)\, ds + g_1(0)\, g_2(0). \tag{4.6}$$

It may be verified that, for this example,

$$P_{T_n} Q_t = \frac{(t_{i+1} - t)}{(t_{i+1} - t_i)} Q_{t_i} + \frac{(t - t_i)}{(t_{i+1} - t_i)} Q_{t_{i+1}}, \qquad t \in [t_i, t_{i+1}]. \tag{4.7}$$

We note that (4.7) implies that minimum norm interpolation in $\mathscr{H}_Q$ is linear interpolation, that is,

$$\begin{aligned} \hat{g}(t) &= P_{T_n} g(t) = \langle P_{T_n} g, Q_t \rangle_Q = \langle g, P_{T_n} Q_t \rangle_Q \\ &= \frac{(t_{i+1} - t)}{(t_{i+1} - t_i)} g(t_i) + \frac{(t - t_i)}{(t_{i+1} - t_i)} g(t_{i+1}), \qquad t \in [t_i, t_{i+1}]. \end{aligned} \tag{4.8}$$

Since

$$L_m \hat{f}(t) = \hat{g}(t), \qquad \hat{f} \in \mathscr{B}, \tag{4.9}$$

where $\hat{f}(t)$ is given by (2.10), we have, exactly

$$\hat{f}(t) = \int_0^1 G_2(t, u)\, \hat{g}(u)\, du \tag{4.10}$$

where $G_2(t, u)$ is the Green's function for the equation

$$f'' = g, f(0) = f(1) = 0,$$

which is being solved approximately.

$$\begin{aligned} G_2(t, u) &= -u(1 - t), \qquad u < t, \\ &= -t(1 - u), \qquad u > t. \end{aligned} \tag{4.11}$$

The approximate solution is thus equivalent to the solution found by interpolating g linearly between t_i and t_{i+1}, and then integrating exactly. In general, the approximate solution is equivalent to the solution found by interpolating g at $t \in \Delta$ in the minimum norm fashion in $\mathscr{H}_Q$, and integrating exactly. We remark, however, that this does *not* imply that the Green's function for the problem

$$L_m f = g, \qquad f \in \mathscr{B},$$

is known; it implies only that

$$L_m \eta_{t_i} = Q_{t_i} \tag{4.12}$$

$$\eta_{t_i} \in \mathscr{B}. \tag{4.13}$$

Equations (4.12) and (4.13) follow from (2.3), (2.11), and the fact that $\eta_{t_i} \in \mathscr{H}_R$.

We next show that the exact error rates of (3.13), (3.14), (3.16), and (3.17) with $q = 1$ here cannot be improved upon.

From (4.10) and (4.11),

$$f(t) - \hat{f}(t) = -(1 - t) \int_0^t u(g(u) - \hat{g}(u))\, du - t \int_t^1 (1 - u)(g(u) - \hat{g}(u))\, du, \tag{4.14}$$

$$f'(t) - \hat{f}'(t) = \int_0^t u(g(u) - \hat{g}(u))\, du - \int_t^1 (1 - u)(g(u) - \hat{g}(u))\, du, \tag{4.15}$$

and,

$$f''(t) - \hat{f}''(t) = g(t) - \hat{g}(t). \tag{4.16}$$

Since $g \in \mathscr{H}_Q$, $g' \in \mathscr{L}_2[0, 1]$ and we may write

$$g(t) = g(t_i) + \int_{t_i}^{t} g'(u)\, du, \qquad t \geqslant t_i, \tag{4.17}$$

and hence, using (4.8),

$$\begin{aligned} &| f''(t) - \hat{f}''(t)| \\ &\quad = | g(t) - \hat{g}(t)| = \left| \frac{(t_{i+1} - t)}{(t_{i+1} - t_i)} \int_{t_i}^{t} g'(u)\, du - \frac{(t - t_i)}{(t_{i+1} - t_i)} \int_{t}^{t_{i+1}} g'(u)\, du \right| \\ &\quad \leqslant (t_{i+1} - t_i)^{1/2} \left[\int_{t_i}^{t_{i+1}} (g'(u))^2\, du\right]^{1/2} = O(\| \Delta \|^{q-1/2}), \qquad t \in [t_i, t_{i+1}]. \end{aligned} \tag{4.18}$$

Thus,

$$\int_{t_i}^{t_{i+1}} |(g(u) - \hat{g}(u))|\, du \leqslant (t_{i+1} - t_i)^{3/2} \left[\int_{t_i}^{t_{i+1}} (g'(u))^2\, du\right]^{1/2},$$

and so

$$\begin{aligned} | f'(t) - \hat{f}'(t)| &\leqslant \sum_i (t_{i+1} - t_i)^{3/2} \left[\int_{t_i}^{t_{i+1}} (g'(u))^2\, du\right]^{1/2}, \\ &\leqslant \left(\sum_i (t_{i+1} - t_i)^3\right)^{1/2} \left(\int_0^1 (g'(u))^2\, du\right)^{1/2}, \\ &= O \| \Delta \|^q, \end{aligned} \tag{4.19}$$

and similarly

$$| f(t) - \hat{f}(t)| = O\| \Delta \|^q. \tag{4.20}$$

Equations (4.18), (4.19), and (4.20) thus agree with (3.13) and (3.14). If g' is allowed to be sufficiently badly behaved, then no faster convergence can obtain.

If g has a representation

$$g(t) = \int_0^1 Q(t, t')\, \rho(t')\, dt' \tag{4.21}$$

with $\rho \in \mathscr{L}_2[0, 1]$ then we may write

$$g(t) - \hat{g}(t) = \int_{t_i}^{t_{i+1}} B_i(t, u)\, g''(u)\, du, \qquad t \in [t_i, t_{i+1}], \tag{4.22}$$

where

$$B_i(t, u) = (t_{i+1} - t)(u - t_i)/(t_{i+1} - t_i), \qquad u < t, \tag{4.23}$$

$$= (t_{i+1} - u)(t - t_i)/(t_{i+1} - t_i), \qquad t > u. \tag{4.24}$$

This follows since $g(t) - \hat{g}(t) = 0$, for $t = t_1, t_2, ..., t_n$, and $\hat{g}''(t) = 0$ a.e.

Then

$$|f''(t) - \hat{f}''(t)| = |g(t) - \hat{g}(t)| \leqslant \left[\int_{t_i}^{t_{i+1}} B_i^2(t, u)\, du \cdot \int_{t_i}^{t_{i+1}} (g''(u))^2\, du\right]^{1/2},$$

$$= \frac{1}{\sqrt{3}} \frac{(t_{i+1} - t)(t - t_i)}{(t_{i+1} - t_i)^{1/2}} \left[\int_{t_i}^{t_{i+1}} (g''(u))^2\, du\right]^{1/2}$$

$$= O(\| \Delta \|^{2q-1/2}), \tag{4.25}$$

and, if g'' is allowed to be sufficiently badly behaved, then no faster convergence can obtain. Equation (4.25) agrees with (3.17).

If g'' bounded, then

$$|f''(t) - \hat{f}''(t)| = |g(t) - \hat{g}(t)| \leqslant \max_u |g''(u)| \int_{t_i}^{t_{i+1}} B_i(t, u)\, du,$$

$$= \max_u |g''(u)|\, (t_{i+1} - t)(t - t_i)/2,$$

$$= O(\| \Delta \|^{2q}), \tag{4.26}$$

which is a faster rate than that given by (3.17). However

$$\int_{t_i}^{t_{i+1}} |(g(t) - \hat{g}(t))\, dt\,| = \int_{t_i}^{t_{i+1}} dt \left| \int_{t_i}^{t_{i+1}} B_i(t, u)\, g''(u)\, du \right|,$$

$$\leqslant \max_u |g''(u)| \int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t_{i+1}} B_i(t, u)\, du,$$

$$= \max_u |g''(u)|\, [(t_{i+1} - t_i)^3/2 \cdot 3!], \tag{4.27}$$

and so

$$|f'(t) - \hat{f}'(t)| \leqslant \max_u |g''(u)| \cdot \frac{1}{2 \cdot 3!} \sum_i (t_{i+1} - t_i)^3 = O(\| \Delta \|^{2q}), \tag{4.28}$$

$$|f(t) - \hat{f}(t)| = O(\| \Delta \|^{2q}). \tag{4.29}$$

Equations (4.28) and (4.29) agree with (3.16) and these rates evidently cannot be improved on for g'' bounded, and certainly not for $g'' \in \mathscr{L}_2$. (Take g'' to be a constant over some interval). It appears from the proof in [5] that (2.32) cannot be strengthened for the case ρ bounded, and our methods cannot be used to strengthen (3.17) for this case. Note that, for $\nu < m$ there always exists a g of the form (3.15) so that the Cauchy–Schwartz inequality on the right of (2.28) is an equality. (Take $g = \gamma_s^{\nu}$). However, this cannot be done for $\nu = m$ since γ_s^m is not of the right form.

5. Application to the Approximate Solution of Integrodifferential Equations

Consider the equation

$$\int_0^1 F(t, u) f(u)\, du + L_m f(t) = g(t), \qquad f \in \mathscr{B}, \tag{5.1}$$

where F is a Hilbert–Schmidt kernel, L_m and $\mathscr{B}$ are as in Section 3, and suppose $g \in C^{q-1}$, $g^{(q)} \in \mathscr{L}_2[0, 1]$, and, without loss of generality, suppose $\| FG_m \| < 1$, where G_m is defined by

$$\begin{aligned} f &= G_m g, \\ f(t) &= \int_0^1 G_m(t, u)\, g(u)\, du, \end{aligned} \tag{5.2}$$

G_m being the Green's function of Section 3.

Then, we may write (actually for $g \in \mathscr{L}_2[0, 1]$)

$$f = Mg, \tag{5.3}$$

where M is the Hilbert–Schmidt operator

$$M = G_m(I - FG_m + (FG_m)^2 - (FG_m)^3 + \cdots). \tag{5.4}$$

We seek an approximate solution in $\mathscr{H}_R$, where $\mathscr{H}_R$ may be chosen as in Section 3, based on the assumed properties of the solution f.

Then (5.1) may be written

$$\langle \eta_t, f \rangle_R = g(t), \qquad t \in T, \qquad f \in \mathscr{H}_R, \tag{5.5}$$

where

$$\eta_t(s) = \int_0^1 F(t, u)\, R(s, u)\, du + \sum_{j=0}^{m} a_{m-j}(t) \frac{\partial^j}{\partial t^j} R(s, t). \tag{5.6}$$

Observing that

$$Q(t, t') = \langle \eta_t , \eta_{t'} \rangle_R = \int_0^1 \int_0^1 F(t, u) R(u, v) F(t', v)\, du\, dv$$

$$+ \int_0^1 F(t, u) \sum_{j=0}^{m} a_{m-j}(t') \frac{\partial^j}{\partial t'^j} R(u, t')\, du$$

$$+ \int_0^1 \sum_{j=0}^{m} a_{m-j}(t) \frac{\partial^j}{\partial t^j} R(t, v) F(t', v)\, dv$$

$$+ \sum_{j=0}^{m} \sum_{k=0}^{m} a_{m-j}(t)\, a_{m-k}(t') \frac{\partial^{j+k}}{\partial t^j\, \partial t'^k} R(t, t'), \quad (5.7)$$

an approximate solution $\hat{f}(s)$ is then defined by (2.10) with $\eta_t(s)$ and $Q(t, t')$ given by (5.6) and (5.7).

To use (2.28) and (2.32) to obtain convergence rates for $| f^{(\nu)}(s) - \hat{f}^{(\nu)}(s)|$, we need an expression for γ_s^ν, the element in $\mathscr{H}_Q$ corresponding to R_s^ν under the isomorphism of (2.19). Following the reasoning of (3.20)–(3.22), we use, for $f \in \mathscr{H}_R \sim g \in \mathscr{H}_Q$,

$$\langle \gamma_s , g \rangle_Q = \langle R_s , f \rangle_R = f(s) = \int_0^1 M(s, u)\, g(u)\, du, \qquad (5.8)$$

where $M(s, u)$ is the Hilbert–Schmidt kernel for M of (5.4). Thus

$$\gamma_s(t) = \langle \gamma_s , Q_t \rangle_Q = \int_0^1 M(s, u)\, Q_t(u)\, du, \qquad (5.9)$$

$$\gamma_s^\nu(t) = \int_0^1 Q(t, u)\, \psi_s^\nu(u)\, du, \qquad \nu = 0, 1, 2, \ldots, m - 1, \qquad (5.10)$$

where

$$\psi_s^\nu(u) = (\partial^\nu / \partial s^\nu)\, M(s, u). \qquad (5.11)$$

If $F(t, u)$ is sufficiently smooth, then $Q(t, t')$ will satisfy hypotheses (i) and (ii) of Theorem 1, and ψ_s^ν will be piece wise continuous, $\nu = 0, 1, \ldots, m - 1$. In this case $\| \gamma_s^\nu - P_{T_n} \gamma_s^\nu \|_Q = O(\| \Delta \|^q)$ and hence (3.13) holds; if further g satisfies (3.15), then (3.16) holds.

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